Mathematical Modelling Exam

August 18th, 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times r}$ and $C \in \mathbb{R}^{m\times r}$ be matrices. Consider the solutions of the matrix equations:
$$AXB = C. \tag{1}$$
Let $G_1 \in \mathbb{R}^{n\times m}$ and $G_2 \in \mathbb{R}^{r\times m}$ be generalized inverses of A and B, respectively.
   (a) Assume that $C = AG_1CG_2B$. Check that G_1CG_2 solves (1).
   (b) Prove that if (1) is solvable, then $C = AG_1CG_2B$ holds.
   Hint: Multiply (1) from left and from right by appropriate matrices and use the definitions of G_1, G_2.
   (c) Assume that (1) is solvable. Check that
$$X = G_1CG_2 + Z - G_1AZBG_2$$
   solves (1) for any $Z \in \mathbb{R}^{n\times p}$.

2. Let
$$\sqrt{\pi}\ln(x_1^2+x_2^2) - \frac{1}{\sqrt{\pi}}\sin(x_1x_2) = \ln(2\pi),$$
$$e^{x_1-x_2} + \frac{1}{\sqrt{\pi}}\cos(x_1x_2) = 0,$$
be a nonlinear system and $v^{(0)} = \begin{bmatrix} \sqrt{\pi} & \sqrt{\pi} \end{bmatrix}^T$ a vector. Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.

3. Sketch the curve given in polar coordinates by
$$r(\varphi) = 2 + 4\sin(\varphi)$$
and compute the area of the smaller bounded region determined by the curve.

4. Solve the differential equation
$$\ddot{x} - \dot{x} - 4x = 2t + e^t. \tag{2}$$