Mathematical modelling

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1. We are given four points: $(0,1),(-1,0),(1,2),(2,3)$. We would like to fit a function of the form ax^2+bx to these points.
   (a) Write down the matrix A of the corresponding system of linear equations.
   (b) Find the Moore-Penrose inverse A^+.
   (c) Find the function of the above form that fits the points best according to the least squares criterion.
   (d) Find one more generalized inverse of A.

   Solution.

   (a) The matricial form of the system is the following:

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 1 \\ 4 & 2 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}}_{c}.$$

   (b) Since $\operatorname{rank} A = 2$, also $\operatorname{rank}(A^TA)=2$ and hence $A^\dagger$ is equal to

$$\begin{aligned} A^\dagger = (A^TA)^{-1}A^T &= \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{22} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{9}{22} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{7}{22} & -\frac{1}{22} & \frac{2}{11} \\ 0 & -\frac{13}{22} & \frac{5}{22} & \frac{1}{11} \end{bmatrix}. \end{aligned}$$

   (c) The solution a,b such that ax^2+bx fits the data best w.r.t. the least squares error method is

$$A^\dagger c = \begin{bmatrix} 0 & \frac{7}{22} & -\frac{1}{22} & \frac{2}{11} \\ 0 & -\frac{13}{22} & \frac{5}{22} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} \\ \frac{8}{11} \end{bmatrix}.$$

(d) Another generalized inverse of A is

$$G = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}^{-1} \right)^T \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 2 & -\frac{1}{2} \end{bmatrix}.$$

2. Given the parametric curve $\gamma(t) = (t^3 - t + 1, t^2)$:
   (a) Find selfintersections of γ.
   (b) Find the angle at which γ intersects itself in the selfintersections.
   (c) Find the point at which γ reaches its lowest level (smallest y coordinate).

Solution.

(a) Self-intersections are points such that $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$:

$$\begin{aligned} \gamma(t_1) = \gamma(t_2) \quad &\Leftrightarrow \quad t_1^3 - t_1 + 1 = t_2^3 - t_2 + 1 \quad \text{and} \quad t_1^2 = t_2^2, \\ &\Leftrightarrow \quad t_1^3 - t_1 + 1 = -t_1^3 + t_1 + 1 \quad \text{and} \quad t_2 = -t_1, \\ &\Leftrightarrow \quad 2t_1(t_1^2 - 1) \quad \text{and} \quad t_1^2 = t_2^2, \\ &\Leftrightarrow \quad t_1 = 0, t_2 = 0 \quad \text{or} \quad t_1 = -t_2 = 1 \quad \text{or} \quad t_1 = -t_2 = -1. \end{aligned}$$

Hence, the self-intersection is $\gamma(1) = \gamma(-1) = (1, 1)$.

(b) The angle at which γ intersects itself in the self-intersection is the angle between the tangents γ_1, γ_2 in $(1, 1)$:

$$\begin{aligned} \gamma_1(\lambda) &= (1,1) + \lambda \cdot \dot{\gamma}(1) = (1,1) + \lambda \left((3t^2 - 1, 2t)(1) \right) = (1,1) + \lambda(2,2), \\ \gamma_2(\lambda) &= (1,1) + \lambda \cdot \dot{\gamma}(-1) = (1,1) + \lambda \left((3t^2 - 1, 2t)(-1) \right) = (1,1) + \lambda(-4,-2). \end{aligned}$$

Hence,

$$\arccos \left(\frac{\langle (2,2), (-4,-2) \rangle}{\|(2,2)\| \, \|(-4,-2)\|} \right) = \arccos \left(\frac{-8-4}{\sqrt{8}\sqrt{20}} \right) = \arccos \left(\frac{-3}{\sqrt{10}} \right) \approx 2.82.$$

So, the angle between the tangents is $\pi - 2.82 \approx 0.32$.

(c) The point at which γ reaches a a global minimum in the y-directions satisfies $\frac{d}{dt}(t^2) = 0$. Hence, $2t = 0$ and $t = 0$. The point is $(1, 0)$.

3. Solve the differential equation $xy' = y + 2x^3$ with the initial condition $y(2) = 3$.

   Solution. First we solve the homogeneous part of the DE:

$$xy' = y \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln|y| = \ln|x| + k \quad \Rightarrow \quad y_h(x) = Kx,$$

   where $k, K \in \mathbb{R}$ are constants. Now we have to determine one particular solution. By the form of the DE we can try with the form

$$y_p(x) = Ax^3 + Bx^2 + Cx + D,$$

   where $A, B, C, D \in \mathbb{R}$ are constants. Hence,

$$y_p'(x) = 3Ax^2 + 2Bx + C$$

   and plugging into the DE we get

$$x(3Ax^2 + 2Bx + C) = Ax^3 + Bx^2 + Cx + D + 2x^3. \tag{1}$$

   Comparing the coefficients at $x^3, x^2, x, 1$ on both sides of (1) we obtain the system

$$3A = A + 2, \quad 2B = B, \quad C = C, \quad 0 = D.$$

   Hence,

$$A = 1, \quad B = D = 0 \quad \text{and} \quad C \in \mathbb{R} \text{ is arbitrary.}$$

   We choose $C = 0$ and get

$$y_p(x) = x^3.$$

   The general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = Kx + x^3.$$

   The solution which passes through the point $(2, 3)$ is

$$y(2) = 3 = 2K + 8 \quad \Rightarrow \quad K = -\frac{5}{2} \quad \Rightarrow \quad y(x) = -\frac{5}{2}x + x^3.$$

4. Solve the differential equation $y'' + y' - 6y = 36x$. with the initial condition $y(0) = y'(0) = 1$.

Solution. First we solve the homogeneous part of the DE:

$$y'' + y' - 6y = 0.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

with zeroes $\lambda_1 = -3$, $\lambda_2 = 2$. Hence, the solution of the homogeneous part is

$$y_h(x) = Ce^{-3x} + De^{2x},$$

where $C, D \in \mathbb{R}$ are constants.

To obtain a particular solution we can try with the form

$$y_p(x) = ax^2 + bx + c \quad \Rightarrow \quad y_p'(x) = 2ax + b \quad \Rightarrow \quad y_p''(x) = 2a. \quad (2)$$

Plugging (2) into the DE we obtain

$$2a + (2ax + b) - 6(ax^2 + bx + c) = 36x. \quad (3)$$

Comparing the coefficients at $x^2, x, 1$ on both sides of (3) we obtain a system

$$-6a = 0, \quad -6b + 2a = 36, \quad 2a + b - 6c = 0,$$

with the solution

$$a = 0, \quad b = -6, \quad c = -1.$$

Hence, the general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = Ce^{-3x} + De^{2x} - 6x - 1.$$

The one satisfying the initial conditions

$$y(0) = C + D - 1 = 1$$
$$y'(0) = -3C + 2D - 6 = 1,$$

is the one with

$$C = -\frac{3}{5}, \quad D = \frac{13}{5}.$$

So the final solution is

$$y(x) = -\frac{3}{5}e^{-3x} + \frac{13}{5}e^{2x} - 6x - 1.$$